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A characterization of some subsets of Schechter's essential spectrum and application to singular transport equation

Aref Jeribi *, Nedra Moalla

Département de Mathématiques, Laboratoire de Physique-Mathématiques, Université de Sfax, Faculté des Sciences de Sfax, Route de Soukra, Km 3.5, B.P. 1171, 3000 Sfax, Tunisia

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ABSTRACT

The purpose of this paper is to provide a detailed treatment of some subsets of Schechter's essential spectrum of closed, densely defined linear operators subjected to additive perturbations. Our results are used to describe the essential approximate point spectrum and the essential defect spectrum of singular neutron transport operators in bounded geometries.

1. Introduction

Let X and Y be two Banach spaces. We denote by $\mathcal{L}(X,Y)$ (resp. $\mathcal{C}(X,Y)$) the set of all bounded (resp. closed, densely defined) linear operators from X into Y and we denote by $\mathcal{K}(X,Y)$ the subspace of compact operators from X into Y. For $A \in \mathcal{C}(X,Y)$, we write $\mathcal{D}(A) \subset X$ for the domain, $N(A) \subset X$ for the null space and $R(A) \subset Y$ for the range of A. The nullity, $\alpha(A)$, of A is defined as the dimension of $N(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $R(A)$ in Y. Let $\sigma(A)$ (resp. $\rho(A)$) denote the spectrum (resp. the resolvent set) of A. The set of upper semi-Fredholm operators is defined by

$$\Phi_+(X,Y) = \{A \in \mathcal{C}(X,Y) \text{ such that } \alpha(A) < \infty \text{ and } R(A) \text{ is closed in } Y\}$$

and the set of lower semi-Fredholm operators is defined by

$$\Phi_-(X,Y) = \{A \in \mathcal{C}(X,Y) \text{ such that } \beta(A) < \infty \text{ and } R(A) \text{ is closed in } Y\}.$$

$\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y)$ denotes the set of Fredholm operators from X into Y and $\Phi_\pm(X,Y)$: $\Phi_+(X,Y) \cup \Phi_-(X,Y)$ the set of semi-Fredholm operators from X into Y. If $X = Y$ then $\mathcal{L}(X,Y)$, $\mathcal{C}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ and $\Phi_\pm(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi_\pm(X)$, respectively. A complex number λ is in Φ_{+A}, Φ_{-A}, $\Phi_{\pm A}$ or Φ_A if $\lambda - A$ is in $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_\pm(X)$ or $\Phi(X)$, respectively. If A is a semi-Fredholm operator (either upper or lower) the index of A is defined by $i(A) := \alpha(A) - \beta(A)$. Clearly, if $A \in \Phi(X,Y)$ then $i(A) < \infty$. If $A \in \Phi_+(X,Y) \setminus \Phi(X,Y)$, then $i(A) = -\infty$ and if $A \in \Phi_-(X,Y) \setminus \Phi(X,Y)$ then $i(A) = +\infty$.

For a self-adjoint operator in a Hilbert space, there seems to be only one reasonable way to define the essential spectrum: the set of all points of the spectrum that are not isolated eigenvalues of finite algebraic multiplicity (see, for example [29,36,37]). When dealing with a closed, densely defined linear operator, A, on a Banach space X, various notions of essential spectrum appear in applications of spectral theory (see, for instance [6,10,16,30,33,36]) and the references therein.

* Corresponding author.
E-mail address: Aref.Jeribi@fss.rnu.tn (A. Jeribi).

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For $A \in \mathcal{C}(X)$, we introduce the following essential spectra

$$\sigma_{eg}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_+(X)\} := \mathbb{C} \setminus \Phi_{+A},$$
$$\sigma_{ew}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_-(X)\} := \mathbb{C} \setminus \Phi_{-A},$$
$$\sigma_{ess}(A) := \mathbb{C} \setminus \rho_{ess}(A),$$
$$\sigma_{eap}(A) := \bigcap_{K \in \mathcal{K}(X)} \sigma_{ap}(A+K),$$
$$\sigma_{e\delta}(A) := \bigcap_{K \in \mathcal{K}(X)} \sigma_{\delta}(A+K),$$

where

$$\rho_{ess}(A) := \{\lambda \in \Phi_A \text{ such that } i(\lambda - A) = 0\},$$
$$\sigma_{ap}(A) := \Big\{\lambda \in \mathbb{C} \text{ such that } \inf_{\|x\|=1,\, x \in \mathcal{D}(A)} \|(\lambda - A)x\| = 0\Big\},$$
$$\sigma_{\delta}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \text{ is not surjective}\}.$$

The subsets $\sigma_{eg}(\cdot)$ and $\sigma_{ew}(\cdot)$ are the Gustafson and Weidmann essential spectra [9] and $\sigma_{ess}(\cdot)$ denotes the Schechter essential spectrum [30,32]. $\sigma_{eap}(\cdot)$ was introduced by V. Rakočević in [25] and denotes the essential approximate point spectrum and $\sigma_{e\delta}(\cdot)$ is the essential defect spectrum and was introduced by C. Schmoeger [33]. If X is a Hilbert space and A is a self-adjoint operator on X, then all these sets coincide.

When dealing with essential spectra of closed, densely defined linear operators on Banach spaces, one of the main problems consists of studying the invariance of the essential spectra of these operators subjected to various kinds of perturbation. Among the works in this direction we quote, for example, [11–13,30,31]. This work is devoted to this question and continues the works started in [26,33]. Letting $A \in \mathcal{C}(X)$, the question is what are the conditions that we must impose on $K \in \mathcal{C}(X)$ in order that $\sigma_{eap}(A+K) = \sigma_{eap}(A)$ and $\sigma_{e\delta}(A+K) = \sigma_{e\delta}(A)$. If K is a compact operator on the Banach space X, then the result follows from the definitions of $\sigma_{eap}(\cdot)$ and $\sigma_{e\delta}(\cdot)$. If K is an upper semi-Fredholm perturbation (see, Definition 2.5), it is shown in [26] that $\sigma_{eap}(A+K) = \sigma_{eap}(A)$. But in practice, the perturbed operator K is neither compact nor upper semi-Fredholm perturbation (see Section 5). So it is natural to find a larger class of operators, which occur in applications, for which we have the invariance of $\sigma_{eap}(A)$ and $\sigma_{e\delta}(A)$. For this, the aim of this paper consists principally of considering the class of A-bounded operators K (not necessarily bounded) such that $K(\lambda - A)^{-1} \in \mathcal{F}_+^b(X)$ (resp. $((\lambda - A)^{-1}\hat{K})^* \in \mathcal{F}_+^b((X_A)^*)$) for some $\lambda \in \rho(A)$ (where $\mathcal{F}_+^b(X)$ stands for the ideal of upper semi-Fredholm perturbations and X_A denotes the domain of A) and proving that $\sigma_{eap}(A+K) = \sigma_{eap}(A)$ (resp. $\sigma_{e\delta}(A+K) = \sigma_{e\delta}(A)$).

In the last section we study the essential approximate point spectrum and the essential defect spectrum of the following singular neutron transport operator

$$A\psi(x,v) = -v \cdot \nabla_x \psi(x,v) - \sigma(v)\psi(x,v) + \int_{\mathbb{R}^n} \kappa(v,v')\psi(x,v')\,d\mu(v'),$$

with vacuum boundary conditions, i.e., $\psi_{|\Gamma_-} = 0$ with

$$\Gamma_- = \{(x,v) \in \partial D \times \mathbb{R}^n \text{ such that } v \cdot \nu_x < 0\},$$

where ν_x stands for the outer unit normal vector at $x \in \partial D$. Here $(x,v) \in D \times \mathbb{R}^n$, where D is an open bounded set of $\mathbb{R}^n$ with piecewise C^1 boundary and $d\mu(\cdot)$ is a bounded positive measure on $\mathbb{R}^n$. This operator describes the transport of particles (neutrons, photons, molecules of gas, etc.) in the domain D. For the neutrons, the function $\psi(x,v)$ represents the number (or probability) density of gas particles having the position x and the velocity v. For the photons, ψ describes the specific intensity of the light. For the molecules of gas, ψ describes the deviation of the number density of the gas molecules from their equilibrium number density. For gas molecules, the transport equation is obtained by linearization of the nonlinear Boltzmann equation or some nonlinear simplification of it (such as the Enskog equation or the BGK model) about the equilibrium distribution. The functions $\sigma(\cdot)$ and $\kappa(\cdot,\cdot)$ are called, respectively, the collision frequency and the scattering kernel and will be assumed to be unbounded. More precisely, we will assume that there exists a closed subset $\mathcal{O} \subset \mathbb{R}^n$ with zero $d\mu$ measure and a constant $\sigma_0 > 0$ such that

$$\sigma(\cdot) \in L^\infty_{loc}(\mathbb{R}^n \setminus \mathcal{O}), \quad \sigma(v) > \sigma_0 \text{ a.e.} \tag{1.1}$$

and

$$\left[\int_{\mathbb{R}^n} \left(\frac{\kappa(\cdot,v')}{\sigma(v')^{1/p}}\right)^q d\mu(v')\right]^{1/q} \in L_p(\mathbb{R}^n) \tag{1.2}$$

where q denotes the conjugate exponent of p. These assumptions were motivated by free gas models (cf. [22,34]) and were already used by M. Mokhtar-Kharroubi [21] in L_1 spaces and by B. Lods [20] in the case of L_p spaces. The first part of the condition (1.1) means that the singularities of the collision frequency are contained in a set of zero $d\mu$ measure. Actually, unbounded and nonnegative collision frequencies act as strong absorptions which might lead to unbounded collision operators.

We organize our paper in the following way: In Section 2 we gather some results and notations from Fredholm theory connected with Sections 3 and 4. In Section 3 we present a new characterization of the essential approximate point spectrum and the essential defect spectrum. The main results of this section are Theorems 3.1 and 3.2. Section 4 is devoted to the spectral mapping theorem for $\sigma_{eap}(\cdot)$ and $\sigma_{e\delta}(\cdot)$ in a special case which occurs in applications. Finally, in Section 5 we apply the results of Section 3 to investigate the essential approximate point spectrum and the essential defect spectrum of the singular neutron transport operator with vacuum boundary conditions.

2. Preliminaries

Definition 2.1. Let X and Y be two Banach spaces. An operator $A \in \mathcal{L}(X, Y)$ is said to be weakly compact if $A(B)$ is relatively weakly compact in Y for every bounded subset $B \subset X$.

The family of weakly compact operators from X to Y is denoted by $\mathcal{W}(X, Y)$. If $X = Y$ the family of weakly compact operators on X, $\mathcal{W}(X) := \mathcal{W}(X, X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ (cf. [3,5]).

Definition 2.2. Let X and Y be two Banach spaces. An operator $S \in \mathcal{L}(X, Y)$ is called strictly singular if, for every infinite-dimensional subspace M of X, the restriction of S to M is not a homeomorphism.

Let $\mathcal{S}(X, Y)$ denote the set of strictly singular operators from X to Y.

The concept of strictly singular operators was introduced in the pioneering paper by T. Kato [15] as a generalization of the notion of compact operators. For a detailed study of the properties of strictly singular operators we refer to [5,15]. Note that $\mathcal{S}(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$. In general, strictly singular operators are not compact (cf. [4,5]) and if $X = Y$, $\mathcal{S}(X) := \mathcal{S}(X, X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$. If X is a Hilbert space, then $\mathcal{S}(X)= \mathcal{K}(X)$. The class of weakly compact operators in L_1-spaces (resp. $\mathcal{C}(\Omega)$-spaces with Ω a compact Haussdorff space) is nothing but than the family of strictly singular operators on L_1-spaces (resp. $\mathcal{C}(\Omega)$-spaces) (see [24, Theorem 1]).

Let X be a Banach space. If N is a closed subspace of X, we denote by π_N the quotient map $X \to X/N$. The codimension of $N, \operatorname{codim}(N),$ is defined to be the dimension of the vector space X/N.

Definition 2.3. Let X and Y be two Banach spaces. An operator $S \in \mathcal{L}(X, Y)$ is said to be strictly cosingular if there exists no closed subspace N of Y with $\operatorname{codim}(N) = \infty$ such that $\pi_N S : X \to Y/N$ is surjective.

Let $C\mathcal{S}(X, Y)$ denote the set of strictly cosingular operators from X to Y. This class of operators was introduced by A. Pelczynski [24]. It forms a closed subspace of $\mathcal{L}(X, Y)$. If $X = Y$, $C\mathcal{S}(X) := C\mathcal{S}(X, X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ (cf. [35]).

Definition 2.4. A Banach space X is said to have the Dunford–Pettis property (for short property DP) if for each Banach space Y every weakly compact operator $T : X \longrightarrow Y$ takes weakly compact sets in X into norm compact sets of Y.

It is well known that any L_1-space has the property DP [2, Corollary VI.6]. Also, if Ω is a compact Hausdorff space, $C(\Omega)$ has the property DP [8]. For further examples we refer to [1] or [3, pp. 494, 497, 508 and 511]. Note that the property DP is not preserved under conjugation. However, if X is a Banach space whose dual has the property DP then X has the property DP (see, [2, Corollary, p. 177]). For more information we refer to [1,2] which contains a survey and exposition of the Dunford–Pettis property and related topics.

Definition 2.5. Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X, Y)$.

(i) The operator F is called Fredholm perturbation if $U + F \in \Phi(X, Y)$ whenever $U \in \Phi(X, Y)$.
(ii) F is called an upper (resp. lower) semi-Fredholm perturbation if $U + F \in \Phi_+(X, Y)$ (resp. $U + F \in \Phi_-(X, Y)$) whenever $U \in \Phi_+(X, Y)$ (resp. $U \in \Phi_-(X, Y)$).

We denote by $\mathcal{F}(X, Y)$ the set of Fredholm perturbations and by $\mathcal{F}_+(X, Y)$ (resp. $\mathcal{F}_-(X, Y)$) the set of upper semi-Fredholm (resp. lower semi-Fredholm) perturbations.

Remark 2.1. Let $\Phi^b(X,Y)$, $\Phi^b_+(X,Y)$ and $\Phi^b_-(X,Y)$ denote the sets $\Phi(X,Y)\cap\mathcal{L}(X,Y)$, $\Phi_+(X,Y)\cap\mathcal{L}(X,Y)$ and $\Phi_-(X,Y)\cap\mathcal{L}(X,Y)$ respectively. If in Definition 2.5 we replace $\Phi(X,Y)$, $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$ by $\Phi^b(X,Y)$, $\Phi^b_+(X,Y)$ and $\Phi^b_-(X,Y)$ we obtain the sets $\mathcal{F}^b(X,Y)$, $\mathcal{F}^b_+(X,Y)$ and $\mathcal{F}^b_-(X,Y)$.

The sets of Fredholm perturbations and semi-Fredholm perturbations were introduced and investigated in [4]. In particular, it is shown that $\mathcal{F}^b(X,Y)$, $\mathcal{F}^b_+(X,Y)$ and $\mathcal{F}^b_-(X,Y)$ are closed subsets of $\mathcal{L}(X,Y)$ and if $X=Y$, then $\mathcal{F}^b(X):=\mathcal{F}^b(X,X)$, $\mathcal{F}^b_+(X):=\mathcal{F}^b_+(X,X)$ and $\mathcal{F}^b_-(X):=\mathcal{F}^b_-(X,X)$ are closed two-sided ideals of $\mathcal{L}(X)$.

In general, we have the following inclusions

$$\mathcal{K}(X,Y)\subset\mathcal{S}(X,Y)\subset\mathcal{F}^b_+(X,Y)\subset\mathcal{F}^b(X,Y),\tag{2.1}$$

$$\mathcal{K}(X,Y)\subset C\mathcal{S}(X,Y)\subset\mathcal{F}^b_-(X,Y)\subset\mathcal{F}^b(X,Y).\tag{2.2}$$

The inclusion $\mathcal{S}(X,Y)\subset\mathcal{F}^b_+(X,Y)$ is due to T. Kato [15], whereas the inclusion $C\mathcal{S}(X,Y)\subset\mathcal{F}^b_-(X,Y)$ was proved by J.I. Vladimirskii [35].

Lemma 2.1. *Let $A\in\mathcal{C}(X,Y)$ and $F\in\mathcal{L}(X,Y)$. Then*

(i) *If $A\in\Phi^b(X,Y)$ and $F\in\mathcal{F}^b(X,Y)$, then $A+F\in\Phi^b(X,Y)$ and $i(A+F)=i(A)$.*
(ii) *If $A\in\Phi^b_+(X,Y)$ and $F\in\mathcal{F}^b_+(X,Y)$, then $A+F\in\Phi^b_+(X,Y)$ and $i(A+F)i(A)$.*
(iii) *If $A\in\Phi^b_-(X,Y)$ and $F\in\mathcal{F}^b_-(X,Y)$, then $A+F\in\Phi^b_-(X,Y)$ and $i(A+F)i(A)$.*

Proof. Statement (i) follows from [4, Proposition 3].

(ii) Let $A\in\Phi^b_+(X,Y)$ and $F\in\mathcal{F}^b_+(X,Y)$. The fact that $A+F\in\Phi^b_+(X,Y)$ follows from Definition 2.5. To prove that $i(A+F)=i(A)$, we discuss two cases.

Case 1: If $A\in\Phi^b_+(X,Y)\setminus\Phi^b(X,Y)$, then $i(A)=-\infty$. Hence, $i(A+F)=-\infty$. Otherwise, $A+F\in\Phi^b(X,Y)$ and therefore $A\in\Phi^b_+(X,Y)$, since $F\in\mathcal{F}^b_+(X,Y)\subset\mathcal{F}^b(X,Y)$. What is contradictory.

Case 2: If $A\in\Phi^b(X,Y)$, then the result follows from the assertion (i).

Statement (iii) can be checked in the same way as (ii). □

Remark 2.2. The result of this lemma remains valid if we replace $\Phi^b(X,Y)$ by $\Phi(X,Y)$ and $\mathcal{F}^b(X,Y)$ by $\mathcal{F}(X,Y)$.

3. Main results

The purpose of this section is to discuss the essential approximate point spectrum and the essential defect spectrum of a closed, densely defined linear operator on a Banach space X. We begin with the following useful result.

Lemma 3.1. *Let $A\in\Phi_+(X)$. Then the following statements are equivalent*

(i) $i(A)\leqslant 0$;
(ii) *A can be expressed in the form $A=U+K$ where $K\in\mathcal{K}(X)$ and $U\in\mathcal{C}(X)$ an operator with closed range and $\alpha(U)=0$.*

This lemma is well known for bounded upper semi-Fredholm operators. The proof is a straightforward adaption of the proof of Theorem 3.9 in [38].

The results of the next proposition was established in [25] and [33] for bounded linear operators. We will improve it for closed, densely defined linear operators. This result is a characterization of the essential approximate point spectrum (resp. the essential defect spectrum) by means of upper semi-Fredholm (resp. lower semi-Fredholm) operators.

Proposition 3.1. *Let $A\in\mathcal{C}(X)$, then*

(i) *$\lambda\notin\sigma_{eap}(A)$ if and only if $\lambda-A\in\Phi_+(X)$ and $i(\lambda-A)\leqslant 0$.*
(ii) *$\lambda\notin\sigma_{e\delta}(A)$ if and only if $\lambda-A\in\Phi_-(X)$ and $i(\lambda-A)\geqslant 0$.*
(iii) *If A is a bounded linear operator, then $\sigma_{e\delta}(A)=\sigma_{eap}(A^*)$, where A^* stands for the adjoint operator.*

Proof. (i) Let $\lambda\in\Phi_{+A}$ such that $i(\lambda-A)\leqslant 0$. Then by Lemma 3.1, $\lambda-A$ can be expressed in the form $\lambda-A=U+K$ where $K\in\mathcal{K}(X)$ and $U\in\mathcal{C}(X)$ an operator with closed range and $\alpha(U)=0$. Hence by [32, Theorem 5.1, p. 70] there exists a constant $c > 0$ such that $\|Ux\| \geqslant c\|x\|$, for all $x \in \mathcal{D}(A)$. Thus $\lambda \notin \sigma_{ap}(A + K)$ and therefore $\lambda \notin \sigma_{eap}(A)$. Conversely, if $\lambda \notin \sigma_{eap}(A)$, then there exists $K \in \mathcal{K}(X)$ such that

$$\inf_{\|x\|=1,\, x\in\mathcal{D}(A)} \big\|(\lambda - A - K)x\big\| > 0.$$

The use of [32, Theorem 5.1, p. 70] leads to $\lambda - A - K \in \Phi_+(X)$ and $\alpha(\lambda - A - K) = 0$, hence it follows from Lemma 2.1 that $\lambda - A \in \Phi_+(X)$ and $i(\lambda - A) \leqslant 0$. This completes the proof of (i).

The proof of (ii) is a straightforward adoption of the proof of Proposition 7 in [33].

(iii) This assertion follows, immediately, from (i) and (ii). □

Remark 3.1. It follows, immediately, from Proposition 3.1 that, for $A \in \mathcal{C}(X)$

(i) $\sigma_{ess}(A) = \sigma_{eap}(A) \cup \sigma_{e\delta}(A)$.
(ii) $\sigma_{eg}(A) \subset \sigma_{eap}(A)$ and $\sigma_{ew}(A) \subset \sigma_{e\delta}(A)$.

Lemma 3.2. *Let $A \in \mathcal{C}(X)$. Then*

(i) $\sigma C(A) \subset \sigma_{eap}(A) \cap \sigma_{e\delta}(A)$,
(ii) $\sigma R(A) \subset \sigma_{e\delta}(A)$,

where $\sigma C(\cdot)$ (resp. $\sigma R(\cdot)$) stands for the continuous spectrum (resp. the residual spectrum).

Proof. Let $\lambda \in \sigma C(A)$, then $R(\lambda - A)$ is not closed, otherwise $\lambda \in \rho(A)$. Thus, by Proposition 3.1 $\lambda \in \sigma_{eap}(A) \cap \sigma_{e\delta}(A)$, this proves (i). Consider $\lambda \in \sigma R(A)$, then $\beta(\lambda - A) \neq 0$, hence $i(\lambda - A) < 0$, since $\lambda - A$ is one to one. This implies, by the use of Proposition 3.1(ii), that $\lambda \in \sigma_{e\delta}(A)$. □

Let $A \in \mathcal{C}(X)$. Since A is closed, one can make $\mathcal{D}(A)$ into a Banach space by equipping it with the graph norm $\|.\|_A$ (i.e., $\|x\|_A := \|x\| + \|Ax\|$). In this new space, denoted by X_A, the operator A satisfies $\|Ax\| \leqslant \|x\|_A$, and consequently, $A \in \mathcal{L}(X_A, X)$. Let J be a linear operator on X. If $\mathcal{D}(A) \subset \mathcal{D}(J)$, then J will be called A-defined, its restriction to $\mathcal{D}(A)$ will be denoted by $\hat{J}$. Moreover, if $\hat{J} \in \mathcal{L}(X_A, X)$, we say that J is A-bounded. One checks easily that if J is closed (or closable), then J is A-bounded (see [16, Remark 1.5, p. 191]).

Remark 3.2. We say that an operator J is A-closed if $x_n \to x$, $Ax_n \to y$, $Jx_n \to z$, for $\{x_n\} \subseteq \mathcal{D}(A)$ implies that $x \in \mathcal{D}(J)$ and $Jx = z$. It will be called A-closable if $x_n \to 0$, $Ax_n \to 0$, $Jx_n \to z$ implies $z = 0$. It is evident that if J is closed (resp. closable), then J is A-closed (resp. A-closable). Note, however, if A is closed, by [31, Lemma 2.1] we get the equivalence between the following three concepts: (i) J is A-closed, (ii) J is A-closable, (iii) J is A-bounded.

Let J be an arbitrary A-bounded operator. Hence we can regard A and J as operators from X_A into X. They will be denoted by $\hat{A}$ and $\hat{J}$ respectively. These belong to $\mathcal{L}(X_A, X)$. Furthermore, we have the obvious relations

$$\begin{cases} \alpha(\hat{A}) = \alpha(A), \quad \beta(\hat{A}) = \beta(A), \quad R(\hat{A}) = R(A), \\ \alpha(\hat{A} + \hat{J}) = \alpha(A + J), \\ \beta(\hat{A} + \hat{J}) = \beta(A + J) \quad \text{and} \quad R(\hat{A} + \hat{J}) = R(A + J). \end{cases} \tag{3.1}$$

Let $A \in \mathcal{C}(X)$ with a non-empty resolvent set. We define the upper spectrum of A by

$$\sigma_+(A) = \bigcap_{K\in\mathcal{D}_A(X)} \sigma_{ap}(A + K)$$

where $\mathcal{D}_A(X) := \{K \in \mathcal{C}(X)$ such that K is A-bounded and $K(\mu - A)^{-1} \in \mathcal{F}_+^b(X)$, for some $\mu \in \rho(A)\}$.

We also, define the lower spectrum of A by

$$\sigma_-(A) = \bigcap_{K\in\mathcal{F}_-(X)} \sigma_\delta(A + K).$$

We are now in the position to express the main result of this section.

Theorem 3.1. *Let X be a Banach space and $A \in \mathcal{C}(X)$ with a non-empty resolvent set. Then*

(i) $\sigma_{eap}(A) = \sigma_+(A)$.
(ii) $\sigma_{e\delta}(A) = \sigma_-(A)$.

Remark 3.3.

(a) It follows, immediately, from Theorem 3.1 that $\sigma_{eap}(A+K)=\sigma_{eap}(A)$ for all $K\in\mathcal{D}_A(X)$ and $\sigma_{e\delta}(A+K)=\sigma_{e\delta}(A)$ for all $K\in\mathcal{F}_-(X)$.
(b) We can deduce from [4, pp. 69–70] and (2.1) that

$$\mathcal{K}(X)\subset\mathcal{S}(X)\subset\mathcal{F}_+^b(X)\subset\mathcal{D}_A(X).$$

This proves the stability of $\sigma_{eap}(\cdot)$ by means of strictly singular operators and upper semi-Fredholm perturbations.
(c) It is proved in [18, Section 3] that if X is a Banach space with the property DP, then

$$\mathcal{W}(X)\subset\mathcal{F}_+(X)\cap\mathcal{F}_-(X).$$

Thus the spectra $\sigma_{eap}(\cdot)$ and $\sigma_{e\delta}(\cdot)$ are invariant under weakly compact perturbations on this class of Banach spaces.
(d) $\mathcal{K}(X)$ is the minimal subset of $\mathcal{C}(X)$ (in the sense of inclusion) which characterize the essential approximate point spectrum and the essential defect spectrum. Hence Theorem 3.1 provides an improvement of the definition of $\sigma_{eap}(\cdot)$ and $\sigma_{e\delta}(\cdot)$ valid for a somewhat large variety of subsets of $\mathcal{C}(X)$. Then it may be viewed as an extension of [27, Theorem 3.6] and [33, p. 173].
(e) Let $A\in\mathcal{C}(X)$. If $\sigma_{e\delta}(A)\neq\emptyset$, then for all $K\in\mathcal{F}_-(X)$

$$\sigma(A+K)=\sigma P(A+K),$$

where $\sigma P(\cdot)$ stands for the point spectrum.

Proof of Theorem 3.1. (i) Since $\mathcal{K}(X)\subset\mathcal{D}_A(X)$, we infer that $\sigma_+(A)\subset\sigma_{eap}(A)$. Conversely, let $\lambda\notin\sigma_+(A)$, then there exists $K\in\mathcal{D}_A(X)$ such that

$$\inf_{\|x\|=1,\,x\in\mathcal{D}(A)}\big\|(\lambda-A-K)x\big\|>0.$$

The use of [32, Theorem 5.1, p. 70] makes us conclude that $\lambda-A-K\in\Phi_+(X)$. Since $Y:=R(\lambda-A-K)$ is a closed subspace of X, then Y itself is a Banach space with the same norm. Therefore, $(\lambda-\hat{A}-\hat{K})^{-1}\in\mathcal{L}(Y,X_A)$. Let $\mu\in\rho(A)$ such that $K(\mu-A)^{-1}\in\mathcal{F}_+^b(X)$, then we have

$$\hat{K}(\lambda-\hat{A}-\hat{K})^{-1}=\hat{K}(\mu-\hat{A})^{-1}\big[\mathcal{I}+(\mu-\lambda+\hat{K})(\lambda-\hat{A}-\hat{K})^{-1}\big] \tag{3.2}$$

where $\mathcal{I}$ denotes the embedding operator which maps every $x\in Y$ onto the same element in X. Since $(\mu-\lambda+\hat{K})\in\mathcal{L}(X_A,X)$ and $\hat{K}(\mu-\hat{A})^{-1}\in\mathcal{F}_+^b(X)$, then it follows from [4, p. 70] and Eq. (3.2) that

$$\hat{K}(\lambda-\hat{A}-\hat{K})^{-1}\in\mathcal{F}_+^b(Y,X). \tag{3.3}$$

Clearly, $N(\mathcal{I})=\{0\}$ and $R(\mathcal{I})=Y$. So, $\mathcal{I}\in\Phi_+^b(Y,X)$ and $i(\mathcal{I})\leqslant 0$. Therefore, we can deduce from (3.3) and Lemma 2.1 that

$$\mathcal{I}+\hat{K}(\lambda-\hat{A}-\hat{K})^{-1}\in\Phi_+^b(Y,X)\quad\text{and}\quad i\big(\mathcal{I}+\hat{K}(\lambda-\hat{A}-\hat{K})^{-1}\big)\leqslant 0. \tag{3.4}$$

Thus writing $\lambda-\hat{A}$ in the form

$$\lambda-\hat{A}=\big(\mathcal{I}+\hat{K}(\lambda-\hat{A}-\hat{K})^{-1}\big)(\lambda-\hat{A}-\hat{K})$$

and using (3.4) together with [23, Theorem 5, p. 150] and [23, Theorem 12, p. 152] we get $\lambda-\hat{A}\in\Phi_+^b(X_A,X)$ and $i(\lambda-\hat{A})\leqslant 0$. Now, using (3.1) we infer that $\lambda-A\in\Phi_+(X)$ and $i(\lambda-A)\leqslant 0$. Finally, the use of Proposition 3.1 shows that $\lambda\notin\sigma_{eap}(A)$, this proves the assertion (i).

(ii) Since $\mathcal{K}(X)\subset\mathcal{F}_-(X)$ [17, Remark 2.4], then $\sigma_-(A)\subset\sigma_{e\delta}(A)$. It remains to show that $\sigma_{e\delta}(A)\subset\sigma_-(A)$. To do this, we consider $\lambda\notin\sigma_-(A)$, then there exists $F\in\mathcal{F}_-(X)$ such that $\lambda\notin\sigma_\delta(A+F)$. Thus $\lambda-A-F$ is surjective, hence $\lambda-A-F\in\Phi_-(X)$ and $i(\lambda-A-F)=\alpha(\lambda-A-F)\geqslant 0$. Therefore, by Lemma 2.1(iii) we deduce that $\lambda-A\in\Phi_-(X)$ and $i(\lambda-A)=i(\lambda-A-F)\geqslant 0$. We conclude the proof by using Proposition 3.1. □

Corollary 3.1. *Let X be a Banach space and $\mathcal{M}(X)$ be any subset of $\mathcal{L}(X)$. Then*

(i) *If $\mathcal{K}(X)\subset\mathcal{M}(X)\subset\mathcal{D}_A(X)$, then*

$$\sigma_{eap}(A)=\bigcap_{K\in\mathcal{M}(X)}\sigma_{ap}(A+K).$$

(ii) *If $\mathcal{K}(X)\subset\mathcal{M}(X)\subset\mathcal{F}_-(X)$, then*

$$\sigma_{e\delta}(A)=\bigcap_{K\in\mathcal{M}(X)}\sigma_\delta(A+K).$$

Remark 3.4. It follows, immediately, from Corollary 3.1 that

(i) $\sigma_{eap}(A+K)=\sigma_{eap}(A)$ for all $K\in\mathcal{M}(X)$ such that $\mathcal{K}(X)\subset\mathcal{M}(X)\subset\mathcal{D}_A(X)$.
(ii) $\sigma_{e\delta}(A+K)=\sigma_{e\delta}(A)$ for all $K\in\mathcal{M}(X)$ such that $\mathcal{K}(X)\subset\mathcal{M}(X)\subset\mathcal{F}_-(X)$.

In the next theorem we will give a characterization of $\sigma_{e\delta}(\cdot)$ by means of A-bounded perturbations.

Theorem 3.2. *Let X be a Banach space and $A\in\mathcal{C}(X)$ with a non-empty resolvent set. Then*

$$\sigma_{e\delta}(A)=\bigcap_{K\in\mathcal{H}_A(X)}\sigma_\delta(A+K)$$

where $\mathcal{H}_A(X):=\{K\in\mathcal{C}(X)$ such that K is A-bounded and $((\mu-A)^{-1}\hat{K})^*\in\mathcal{F}_+^b((X_A)^*)$, for some $\mu\in\rho(A)\}$.

Proof. Let $\mathcal{O}:=\bigcap_{K\in\mathcal{H}_A(X)}\sigma_\delta(A+K)$. We infer that $\mathcal{O}\subset\sigma_{e\delta}(A)$. Indeed, if K is a compact operator on X, then $\hat{K}\in\mathcal{K}(X_A,X)$. Hence $((\mu-A)^{-1}\hat{K})^*\in\mathcal{K}((X_A)^*)$, since $(\mu-A)^{-1}\in\mathcal{L}(X,X_A)$. It follows from the fact that $\mathcal{K}((X_A)^*)\subset\mathcal{F}_+^b((X_A)^*)$ that $\mathcal{K}(X)\subset\mathcal{H}_A(X)$. Conversely, let $\lambda\notin\mathcal{O}$, then there exists $K\in\mathcal{H}_A(X)$ such that $\lambda-A-K$ is surjective. Thus $\lambda-A-K\in\Phi_-(X)$ and $\beta(\lambda-A-K)=0$. Therefore, $\lambda-\hat{A}-\hat{K}\in\Phi_-^b(X_A,X)$, then it follows from [23, Theorem 4, p. 150] that $\lambda-(\hat{A})^*-(\hat{K})^*\in\Phi_+^b(X^*,(X_A)^*)$ and $\alpha(\lambda-(\hat{A})^*-(\hat{K})^*)=0$. Reasoning now in the same way as in the proof of Theorem 3.1(i) we deduce that $\lambda-(\hat{A})^*\in\Phi_+^b(X^*,(X_A)^*)$ and $i(\lambda-(\hat{A})^*)\leqslant 0$. This together with Eq. (3.1) allows us to conclude that $\lambda-A\in\Phi_-(X)$ and $i(\lambda-A)\geqslant 0$. Finally, the result follows from Proposition 4.1. □

Note that in applications (transport operators, operators arising in dynamic populations, etc. (see [7,14,28])) we deal with operators A and B such that $B=A+K$ where $A\in\mathcal{C}(X)$ and K is, in general, a closed (or closable) A-defined linear operator. The operator K does not necessarily satisfy the hypotheses of the previous results. For some physical conditions on K, we have information about the operator $(\lambda-A)^{-1}-(\lambda-B)^{-1}(\lambda\in\rho(A)\cap\rho(B))$. So the following useful stability result.

Theorem 3.3. *Let $A,B\in\mathcal{C}(X)$ such that $\rho(A)\cap\rho(B)\neq\emptyset$.*

(i) *If for some $\lambda\in\rho(A)\cap\rho(B)$ the operator $(\lambda-A)^{-1}-(\lambda-B)^{-1}\in\mathcal{F}_+^b(X)$, then*
$$\sigma_{eap}(A)=\sigma_{eap}(B)\quad\textit{and}\quad\sigma_{eg}(A)=\sigma_{eg}(B).$$
(ii) *If for some $\lambda\in\rho(A)\cap\rho(B)$ the operator $(\lambda-A)^{-1}-(\lambda-B)^{-1}\in\mathcal{F}_-^b(X)$, then*
$$\sigma_{e\delta}(A)=\sigma_{e\delta}(B)\quad\textit{and}\quad\sigma_{ew}(A)=\sigma_{ew}(B).$$

Proof. Without loss of generality, we suppose that $\lambda=0$. Hence $0\in\rho(A)\cap\rho(B)$. Therefore, we can write for $\mu\neq0$

$$\mu-A=-\mu\left(\mu^{-1}-A^{-1}\right)A.$$

Since, A is one to one and onto, then

$$\alpha(\mu-A)=\alpha\left(\mu^{-1}-A^{-1}\right)\quad\text{and}\quad R(\mu-A)=R\left(\mu^{-1}-A^{-1}\right).$$

This shows that $\mu\in\Phi_{+A}$ (resp. Φ_{-A}) if and only if $\mu^{-1}\in\Phi_{+A^{-1}}$ (resp. $\Phi_{-A^{-1}}$), in this case we have $i(\mu-A)=i(\mu^{-1}-A^{-1})$. Therefore, it follows from Lemma 2.1 that $\Phi_{+A}=\Phi_{+B}$ (resp. $\Phi_{-A}=\Phi_{-B}$) and $i(\mu-A)=i(\mu-B)$, for each $\mu\in\Phi_{+A}$ (resp. Φ_{-A}), since $A^{-1}-B^{-1}\in\mathcal{F}_+^b(X)$ (resp. $\in\mathcal{F}_-^b(X)$). Hence the use of Proposition 3.1 makes us conclude that $\sigma_{eap}(A)=\sigma_{eap}(B)$ (resp. $\sigma_{e\delta}(A)=\sigma_{e\delta}(B)$). This proves the first part of (i) and (ii). The second part of these assertions follows from [17, Theorem 3.2]. □

4. Spectral mapping theorem

The aim of this section is to discuss a spectral mapping theorem for $\sigma_{eap}(\cdot)$ and $\sigma_{e\delta}(\cdot)$ in a special case which occurs in applications.

The results of the following theorem were established respectively by V. Rakočević [27, Theorem 3.3] and C. Schmoeger [33, Theorem 3] for bounded linear operators. Using the same method as developed in [6, p. 30] we can express the theorem for closed unbounded linear operators. For the convenience of the reader we include a proof.

Theorem 4.1. *Let $A\in\mathcal{C}(X)$ with a non-empty resolvent set, and let f be a complex-valued function that is holomorphic on an open set containing $\sigma(A)\cup\{\infty\}$. Then*

$$\sigma_{eap}\left(f(A)\right)\subseteq f\left(\sigma_{eap}(A)\right)$$

and

$$\sigma_{e\delta}\left(f(A)\right)\subseteq f\left(\sigma_{e\delta}(A)\right).$$

Proof. Let β be a fixed point in $\rho(A)$ and define the function ψ by

$$\psi : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\},$$
$$\lambda \longmapsto \psi(\lambda) = \begin{cases} (\lambda-\beta)^{-1} & \text{if } \lambda \neq \beta, \\ \psi(\beta) = \infty, \\ \psi(\infty) = 0. \end{cases}$$

Let $T = \psi(A)$ and set for $\lambda \neq \beta$, $\mu = \psi(\lambda)$. Writing, $A - \lambda = A - \beta - (\lambda - \beta)$, we obtain

$$(A-\lambda)T = \mu^{-1}(\mu - T), \quad \text{on } X.$$

Since, $R(T) = \mathcal{D}(A-\lambda)$, then

$$R(\mu - T) = R(A - \lambda). \tag{4.1}$$

Also, since T is one-to-one map of X onto $\mathcal{D}(A-\lambda)$, then

$$\alpha(A-\lambda) = \alpha(\mu - T). \tag{4.2}$$

Note, that $0 \in \sigma_{eap}(T)$, because $R(T) = \mathcal{D}(A)$ cannot be closed when A is unbounded. Therefore, using Eqs. (4.1) and (4.2) it is easy to verify that ψ is one-to-one map of $\sigma_{eap}(A)$ onto $\sigma_{eap}(T)$. Define now, the function g by $g(\mu) = fo\psi^{-1}(\mu)$. Then g is holomorphic on a neighborhood $\sigma(T)$ and $g(T) = f(A)$. Hence it follows from [27, Theorem 3.3] that

$$\begin{aligned} \sigma_{eap}\big(f(A)\big) &= \sigma_{eap}\big(g(T)\big) \\ &\subseteq g\big(\sigma_{eap}(T)\big) \\ &= f \circ \psi^{-1}\big(\sigma_{eap}(T)\big) \\ &= f\big(\sigma_{eap}(A)\big). \end{aligned}$$

This proves the result for $\sigma_{eap}(\cdot)$. The result for $\sigma_{e\delta}(\cdot)$ can be checked in the same way. □

Let us recall that the spectral mapping theorem holds true for $\sigma_{eg}(\cdot)$ and $\sigma_{ew}(\cdot)$ (cf. [6]). However, a counter-example given in [26] shows that, in general, it is false for $\sigma_{eap}(\cdot)$. The following result provides a spectral mapping theorem for the essential approximate point spectrum and the essential defect spectrum in a special case which occurs in applications.

Proposition 4.1. *Let A_1 and A_2 be two elements of $\mathcal{C}(X)$ such that $(\lambda - A_1)^{-1} - (\lambda - A_2)^{-1} \in \mathcal{F}_+^b(X)$ (resp. $\in \mathcal{F}_-^b(X)$) for some $\lambda \in \rho(A_1) \cap \rho(A_2)$. If $\sigma_{eap}(A_1) = \sigma_{eg}(A_1)$ (resp. $\sigma_{e\delta}(A_1) = \sigma_{ew}(A_1)$) and if f is a complex-valued function holomorphic on a neighborhood of $\sigma(A_1) \cup \sigma(A_2) \cup \{\infty\}$, then*

$$\sigma_{eap}\big(f(A_k)\big) = f\big(\sigma_{eap}(A_k)\big), \quad k = 1, 2$$

(*resp.*

$$\sigma_{e\delta}\big(f(A_k)\big) = f\big(\sigma_{e\delta}(A_k)\big), \quad k = 1, 2).$$

Proof. For $k = 1$ the result follows from the hypothesis $\sigma_{eap}(A_1) = \sigma_{eg}(A_1)$ and [6, Theorem 7]. For the case $k = 2$, the inclusion $\sigma_{eap}(f(A_2)) \subset f(\sigma_{eap}(A_2))$ follows from Theorem 4.1. It remains to show that $f(\sigma_{eap}(A_2)) \subset \sigma_{eap}(f(A_2))$. To do so, we consider $\lambda \in f(\sigma_{eap}(A_2))$, then there exists $\mu \in \sigma_{eap}(A_2)$ such that $\lambda = f(\mu)$. Hence it follows, from Theorem 3.3 and the hypothesis $\sigma_{eap}(A_1) = \sigma_{eg}(A_1)$, that $\mu \in \sigma_{eg}(A_2)$. Now, applying the spectral mapping theorem for $\sigma_{eg}(\cdot)$ [6, Theorem 7(a)], we obtain $f(\mu) \in \sigma_{eg}(f(A_2)) \subseteq \sigma_{eap}(f(A_2))$. Thus, $\lambda \in \sigma_{eap}(f(A_2))$. This proves the result for $\sigma_{eap}(\cdot)$. The result for $\sigma_{e\delta}(\cdot)$ can be proved in the same way. □

5. Application to transport equation

The aim of this section is the study of the essential approximate point spectrum and the essential defect spectrum of the following singular neutron transport operator

$$A\psi(x, v) = -v \cdot \nabla_x \psi(x, v) - \sigma(v)\psi(x, v) + \int_{\mathbb{R}^n} \kappa(v, v')\psi(x, v')\, d\mu(v'),$$

with vacuum boundary conditions, i.e., $\psi_{|\Gamma_-} = 0$ with

$$\Gamma_- = \big\{(x, v) \in \partial D \times \mathbb{R}^n \text{ such that } v \cdot \nu_x < 0\big\},$$

where ν_x stands for the outer unit normal vector at $x \in \partial D$. Here $(x, v) \in D \times \mathbb{R}^n$, D is an open bounded set of $\mathbb{R}^n$, $d\mu(\cdot)$ is a bounded positive measure on $\mathbb{R}^n$ with piecewise C^1 boundary. This operator describes the transport of particles (neutrons, photons, molecules of gas, etc.) in the domain D. The function $\psi(\cdot,\cdot)$ represents the number (or probability) density of particles having the position x and the velocity v. The functions $\sigma(\cdot)$ and $\kappa(\cdot,\cdot)$ are called, respectively, the collision frequency and the scattering kernel.

The main feature is that the collision frequency $\sigma(\cdot)$ and the collision operators K, where K denotes the integral part of A, are unbounded. Actually, an unbounded collision frequency $\sigma(\cdot)$ acts as a strong absorption which allows the unboundedness of K. We will assume that the scattering kernel $\kappa(\cdot,\cdot)$ is non-negative and there exists a closed subset $\mathcal{O} \subset \mathbb{R}^n$ with zero $d\mu$ measure and a constant $\sigma_0 > 0$ such that

$$\sigma(\cdot) \in L^{\infty}_{loc}\big(\mathbb{R}^n \setminus \mathcal{O}\big), \quad \sigma(v) > \sigma_0 \text{ a.e.} \tag{5.1}$$

and

$$\left[\int_{\mathbb{R}^n} \left(\frac{\kappa(\cdot, v')}{\sigma(v')^{1/p}}\right)^q d\mu(v')\right]^{1/q} \in L_p\big(\mathbb{R}^n\big) \tag{5.2}$$

where q denotes the conjugate exponent of p.

Before going further we first recall the functional setting of the problem: Let

$$X_p := L_p\big(D \times \mathbb{R}^n, dx\, d\mu(v)\big), \quad p > 1,$$
$$X_p^{\sigma} := L_p\big(D \times \mathbb{R}^n, \sigma(v)\, dx\, d\mu(v)\big),$$
$$L_p^{\sigma}\big(\mathbb{R}^n\big) := L_p\big(\mathbb{R}^n, \sigma(v)\, d\mu(v)\big).$$

We define the space W_p by

$$W_p = \{\psi \in X_p \text{ such that } v \cdot \nabla_x \psi \in X_p\}.$$

Next we introduce the following subspace of W_p

$$W_p^0 = \{\psi \in W_p \text{ such that } \psi_{|\Gamma_-} = 0\}.$$

We define the streaming operator T by

$$\begin{cases} T\psi(x, v) = -v \cdot \nabla_x \psi(x, v) - \sigma(v)\psi(x, v), \\ \mathcal{D}(T) = W_p^0 \cap X_p^{\sigma}. \end{cases}$$

The transport operator can be formulated as follows $A = T + K$ where K is the following collision operator

$$K : \psi \longrightarrow K\psi(x, v) := \int_{\mathbb{R}^n} \kappa(v, v')\psi(x, v')\, d\mu(v') \in L_p\big(D \times \mathbb{R}^n\big).$$

We denote by $\widetilde{K}$ the following operator

$$\widetilde{K} : \psi \in L_p\big(\mathbb{R}^n\big) \longrightarrow \widetilde{K}\psi(v) := \int_{\mathbb{R}^n} \kappa(v, v')\psi(v')\, d\mu(v') \in L_p\big(\mathbb{R}^n\big).$$

It follows from the assumption (5.2) that $\widetilde{K} \in \mathcal{L}(L_p^{\sigma}(\mathbb{R}^n), L_p(\mathbb{R}^n))$ and

$$\|\widetilde{K}\|_{\mathcal{L}(L_p^{\sigma}(\mathbb{R}^n), L_p(\mathbb{R}^n))} \leqslant \left\| \left[\int_{\mathbb{R}^n} \left(\frac{\kappa(\cdot, v')}{\sigma(v')^{1/p}}\right)^q d\mu(v')\right]^{1/q} \right\|_{L_p(\mathbb{R}^n)}.$$

Using the boundedness of D we find that $K \in \mathcal{L}(X_p^{\sigma}, X_p)$ with

$$\|K\|_{\mathcal{L}(X_p^{\sigma}, X_p)} \leqslant \left\| \left[\int_{\mathbb{R}^n} \left(\frac{\kappa(\cdot, v')}{\sigma(v')^{1/p}}\right)^q d\mu(v')\right]^{1/q} \right\|_{L_p(\mathbb{R}^n)}. \tag{5.3}$$

Note that a simple calculation using the assumption (5.1) shows that X_p^{σ} is a subset of X_p and the embedding $X_p^{\sigma} \hookrightarrow X_p$ is continuous.

Let $\varphi \in X_p$ and $\lambda \in \mathbb{C}$. We seek ψ in $\mathcal{D}(T)$ satisfying

$$(\lambda - T)\psi = \varphi. \tag{5.4}$$

For $Re\,\lambda > -\sigma_0$, the solution of (5.4) reads as follows

$$\psi(x, v) = \int_0^{t^-(x,v)} e^{-(\lambda+\sigma(v))s}\varphi(x - sv, v)\, ds, \tag{5.5}$$

where $t^-(x,v) = \sup\{t > 0,\ \ x - sv \in D,\ \ 0 < s < t\}$. An immediate consequence of these facts is that $\sigma(T) \subset \{\lambda \in \mathbb{C}$ such that $Re\,\lambda \leqslant -\sigma_0\}$.

Since $\sigma(\cdot)$ is bounded below by σ_0, reasoning similar to [14, Corollary 12.11, p. 272] shows that $\sigma(T) = \{\lambda \in \mathbb{C}$ such that $Re\,\lambda \leqslant -\sigma_0\}$. In fact, by [14, Chapter 12] we can easily check that $\sigma(T)$ is reduced to $\sigma C(T)$ (the continuous spectrum of T). Since $\sigma_{eap}(\cdot)$ and $\sigma_{e\delta}(\cdot)$ are enlargement of the continuous spectrum (see, Lemma 3.2) we infer that

$$\sigma_{e\delta}(T) = \sigma_{eap}(T) = \{\lambda \in \mathbb{C} \text{ such that } Re\,\lambda \leqslant -\sigma_0\}.$$

Lemma 5.1. *The collision operator K is T-bounded.*

Proof. Let $\lambda \in \mathbb{C}$ be such that $Re\,\lambda \geqslant 1$ and consider $\psi \in X_p$. It follows from (5.5) that

$$\int_D \left|(\lambda - T)^{-1}\psi(x,v)\right|^p dx \leqslant \frac{1}{(Re\,\lambda + \sigma(v))^p} \int_D \left|\psi(x,v)\right|^p dx.$$

Therefore,

$$\left\|(\lambda - T)^{-1}\psi\right\|_{X_p^\sigma} \leqslant \sup_{v\in\mathbb{R}^n} \frac{\sigma(v)}{(Re\,\lambda + \sigma(v))^p} \|\psi\|_{X_p}.$$

Hence, $(\lambda - T)^{-1} \in \mathcal{L}(X_p^\sigma, X_p)$. Using now, Eq. (5.3) to deduce that the operator K is T-bounded. $\square$

Lemma 5.2. *(See [19, Proposition 4.1].) Let D be a bounded subset of $\mathbb{R}^n$ and $1 < p < \infty$. If the hypothesis* (5.1) *and* (5.2) *are satisfied, the measure $d\mu$ satisfies*

$$\begin{cases} \text{the hyperplanes have zero } d\mu \text{ measure, i.e.,} \\ \text{for each } e \in S^{n-1},\ d\mu\{v \in \mathbb{R}^n,\ v\cdot e = 0\} = 0, \end{cases}$$

where S^{n-1} denotes the unit sphere of $\mathbb{R}^n$ and the collision operator $K : L_p^\sigma(\mathbb{R}^n) \longrightarrow L_p(\mathbb{R}^n)$ is compact. Then for any $\lambda \in \mathbb{C}$ such that $Re\,\lambda > -\sigma_0$, the operator $K(\lambda - T)^{-1}$ is compact on X_p.

Theorem 5.1. *Assume that the hypotheses of Lemma* 5.2 *are satisfied. Then*

$$\sigma_{eap}(A) = \sigma_{e\delta}(A) = \{\lambda \in \mathbb{C} \text{ such that } Re\,\lambda \leqslant -\sigma_0\}.$$

Proof. It follows from the proof of Lemma 5.1 that for any $\lambda \geqslant 1$, $\|(\lambda - T)^{-1}\|_{\mathcal{L}(X_p, X_p^\sigma)} \leqslant \sup_{v\in\mathbb{R}^n} \frac{\sigma(v)}{(Re\,\lambda+\sigma(v))^p}$. Therefore, since X_p^σ is continuously embedded in X_p, we infer that $\lim_{\lambda\to+\infty} \|K(\lambda - T)^{-1}\|_{\mathcal{L}(X_p)} = 0$. So, there exists $\lambda \in \rho(A)$ such that $r_\sigma(K(\lambda - T)^{-1}) < 1$. For such λ we have

$$(\lambda - A)^{-1} - (\lambda - T)^{-1} = \sum_{n\geqslant 1} (\lambda - T)^{-1}\left[K(\lambda - T)^{-1}\right]^n.$$

Now, the result follows from Theorem 3.3 and Lemma 5.2. $\square$

Remark 5.1. Theorem 5.1 is open for $p = 1$.

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